

A statistical approach for an absorbing growth-fragmentation model

Romain Azaïs* and Alexandre Genadot

Abstract

In the present paper, we focus on semi-parametric methods for estimating the absorption probability and the distribution of the absorbing time of a growth-fragmentation model observed within a long time interval. We establish that the absorption probability is the unique solution in an appropriate space of a Fredholm equation of the second kind whose parameters are unknown. We estimate this important characteristic of the underlying process by solving numerically the estimated Fredholm equation. Even if the study has been conducted for a particular model, our method is quite general.

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1 Introduction

The present paper is dedicated to a statistical approach for a particular growth-fragmentation model $(X_t)_{t \geq 0}$ for which $\Gamma = [0, 1]$ is an absorbing set. The motion of $(X_t)_{t \geq 0}$ involves exponential growth and random jumps at random times. From the observation of only one trajectory of the process starting from $X_0 = x \notin \Gamma$, we propose to estimate the probability of absorption $p(x)$ and the distribution $(t_m(x))_{m \geq 1}$ of the hitting time of Γ . More precisely, $t_m(x)$ is the probability of absorption at the m^{th} jump of the process $(X_t)_{t \geq 0}$ starting from $X_0 = x$. This stochastic process has already been introduced in [16] as a theoretical model for insurance, for which Γ is called *area of poverty*. In [16], the authors establish that the ruin probability $p(x)$ is the solution of an integral equation that they solve numerically. In many aspects, our work and the aforementioned paper are different and complementary. Indeed, we adopt a statistical approach: from the observation of the process within a long time, we propose a semi-parametric procedure for estimating the two main characteristics $p(x)$ and $(t_m(x))_{m \geq 1}$ of this model. In addition, we would like to highlight that naive estimates obtained from the empirical versions of these features would not have worked in the framework of a long time observation.

*Corresponding author. E-mail address: romain.azais@gmail.com

The growth-fragmentation model that we consider is a particular case of piecewise-deterministic Markov processes (PDMP's). The PDMP's were first introduced in the literature by Davis [10] in the eighties as a general class of non-diffusion stochastic models. They form a family of continuous-time Markov processes involving deterministic motion punctuated by random jumps, which occur either in a Poisson-like fashion with nonhomogeneous rate or when the deterministic flow hits the boundary of the state space. The motion of a PDMP depends on three local characteristics namely the jump rate, the flow and the transition kernel. A suitable choice of the state space and these three features provides a large number of stochastic models covering various applications, for example in reliability (see [6, 11]) or in neurosciences (see [5, 14, 19]). These processes have been heavily studied both from theoretical and applied perspectives. One may refer the reader to the recent papers [3, 7, 8] about their ergodicity properties and to [4, 9] for references on optimal control and stopping for this class of stochastic models. There are only a few papers which deal with statistical methods for PDMP's. Without attempting to make an exhaustive survey about these methods, we refer the reader to [1, 2, 12, 13, 15] and the references therein. The authors of [12, 13] focus on estimation procedures for ergodic size-structured models, while in [1, 2], the authors are interested in the nonparametric estimation of the transition measure and of the conditional distribution of the inter-jumping times for a general PDMP observed within a long time, under some ergodicity conditions. We would like to emphasize that these methods should not work in our non-ergodic framework. In the book [15], the author computes likelihood processes for observation of PDMP's without boundary jumps. This approach could lead to some estimation methods in the parametric case.

Our framework is semi-parametric. Indeed, the features of the PDMP that we consider are defined from a density function G on $[0, 1]$ and a parameter $\lambda > 0$. Our approach consists in building estimators of the probability of absorption and of the distribution of the absorbing time from some estimators of the features G and λ . For this purpose, we show that the probability of absorption is solution of a Fredholm equation of the second kind that we propose to estimate. This allows us to build our estimator of the probability of absorption $p(x)$ as the numerical solution of this estimated equation. Thus, our estimator $\hat{p}_{n,m}(x)$ depends on both the number n of available data and the number m of iterations of the numerical resolution algorithm. This estimator is defined in (18). Therefore, the error is the combination of a statistical error and a numerical error. The result of convergence of $\hat{p}_{n,m}(x)$ towards $p(x)$ is stated in Theorem 3.6. Of course, the absorption probability is an important feature of the model considered, but the law of the hitting time of Γ is a significant complementary information. We also build an estimator of $t_m(x)$, the probability for the process starting from $X_0 = x$, to be absorbed at the m^{th} jump. This estimator, $\hat{t}_{n,m}(x)$, is defined in (20) and the result of convergence of $\hat{t}_{n,m}(x)$ towards $t_m(x)$ is stated in Theorem 3.8. To estimate the two main quantities of interest $p(x)$ and $(t_m(x))_{m \geq 1}$, we need to estimate at first the post-jump location transition kernel of the process $(X_t)_{t \geq 0}$. This estimator is defined in (12) and convergence results are stated in Proposition 3.3.

We take care to illustrate our theoretical results by numerical simulations. For a given example, we carry out the whole estimation procedure for the quantities of interest $p(x)$ and $(t_m(x))_{m \geq 1}$, on a hundred replications and several numbers of observed data (50, 75 and 100 observed jumps). We present the corresponding numerical results in Section 4. Let us notice that these numerical results are quite good despite the low number of data, especially with respect to the sample sizes used in the works [1, 2, 12, 13].

The paper is organized as follows. We begin in Section 2 with the problem formulation. We present the absorbing growth-fragmentation model that we focus on in Subsection 2.1 and the semi-parametric framework that we choose in Subsection 2.2. In addition, two applications of the present stochastic model are presented in Subsection 2.3. Section 3 is devoted to the presentation of our main results. The results about the estimation of the transition kernel of the post-jump locations are presented in Subsection 3.1. Subsections 3.2 and 3.3 gather the two main results of the paper: the construction and convergence of our estimators for the probability and the time of absorption. Our results are illustrated by numerical simulations within a particular setting in Section 4. Some additional results and the proofs of the main results have been deferred to Appendix A, B and C.

2 Problem formulation

In this section, we present the absorbing growth-fragmentation model that we focus on and some applications. In addition, we present the semi-parametric framework that we choose for investigating the statistical inference for this process observed within a long time interval.

2.1 A growth-fragmentation model

The growth-fragmentation model that we consider is the Markov process $(X_t)_{t \geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ – from a probability density function G on the interval $[0, 1]$ and two real numbers $r, \lambda > 0$ – by its extended generator \mathcal{A} as follows,

$$\mathcal{A}f(x) = r(x-1)^+ f'(x) + \lambda \int_0^1 (f(zx) - f(z)) G(z) dz \quad (1)$$

with $\xi^+ = \xi \vee 0$, and for any smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ (see [10] for full details on the domain of \mathcal{A}). Some applications of this stochastic model are presented in Subsection 2.3. Let us notice that this process has already been introduced as a stochastic model for insurance in [16]. In the sequel, we propose to describe the motion of our growth-fragmentation model as the dynamic of a piecewise-deterministic Markov process (PDMP), see the book [10] and the references therein.

In most cases, the dynamic of a one-dimensional real-valued PDMP is described by its three local features $(\tilde{\lambda}, \mathcal{Q}, \Phi)$.

- $\Phi : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is the deterministic flow. It satisfies,

$$\forall \xi \in \mathbb{R}, \forall s, t \geq 0, \Phi(\xi, t+s) = \Phi(\Phi(\xi, t), s).$$

- $\tilde{\lambda} : \mathbb{R} \rightarrow \mathbb{R}_+$ is the jump rate. It should satisfy,

$$\forall \xi \in \mathbb{R}, \exists \varepsilon > 0, \int_0^\varepsilon \tilde{\lambda}(\Phi(\xi, s)) ds < \infty.$$

- \mathcal{Q} is a Markov kernel on \mathbb{R} which satisfies,

$$\forall \xi \in \mathbb{R}, \mathcal{Q}(\xi, \{\xi\}) = 0.$$

Starting from $X_0 = x$, the motion can be described as follows. The first jump time T_1 is a positive random variable whose survival function is,

$$\forall t \geq 0, \mathbb{P}(T_1 > t | X_0 = x) = \exp \left(- \int_0^t \tilde{\lambda}(\Phi(x, s)) ds \right).$$

This jump time occurs in a Poisson-like fashion with nonhomogeneous rate $\tilde{\lambda}$. One chooses a real-valued random variable Z_1 according to the distribution $\mathcal{Q}(\Phi(x, T_1), \cdot)$. Let us remark that the post-jump location Z_1 depends on the interarrival time T_1 , via the deterministic flow starting from $X_0 = x$. The trajectory between the times 0 and T_1 is given by

$$X_t = \begin{cases} \Phi(x, t) & \text{for } 0 \leq t < T_1, \\ Z_1 & \text{for } t = T_1. \end{cases}$$

Now, starting from X_{T_1} , one may choose the interarrival time $S_2 = T_2 - T_1$ and the post-jump location Z_2 in a similar way as before, and so on. The randomness of such a process is only given by the jump mechanism.

In our particular case, one may easily compute from (1) the local features of the growth-fragmentation model $(X_t)_{t \geq 0}$ that we consider. They are given by

$$\Phi_x(t) = \begin{cases} (x-1) \exp(rt) + 1 & \text{if } x > 1, \\ x & \text{else,} \end{cases} \quad (2)$$

$$\tilde{\lambda}(x) = \lambda \quad \text{and} \quad \mathcal{Q}(x, dy) = \frac{1}{x} G\left(\frac{y}{x}\right) dy. \quad (3)$$

In particular, the rate $\tilde{\lambda}$ is homogeneous. Therefore, the sequence of the interarrival times $(S_n)_{n \geq 1}$ is independent and exponentially distributed with rate λ . In addition, the particular form of the transition kernel \mathcal{Q} (3) implies that the sequence of the random loss fractions $(Y_n)_{n \geq 1}$ defined from,

$$\forall n \geq 1, Z_n = \Phi_{Z_{n-1}}(S_n) Y_n, \quad (4)$$

is independent and independent of $(S_n)_{n \geq 1}$ with common distribution G (see [16]). As a consequence, the dynamic of the PDMP $(X_t)_{t \geq 0}$ may be summarized by the observation of the independent sequences $(S_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$. Two possible trajectories of the process $(X_t)_{t \geq 0}$ are given in Figure 1.

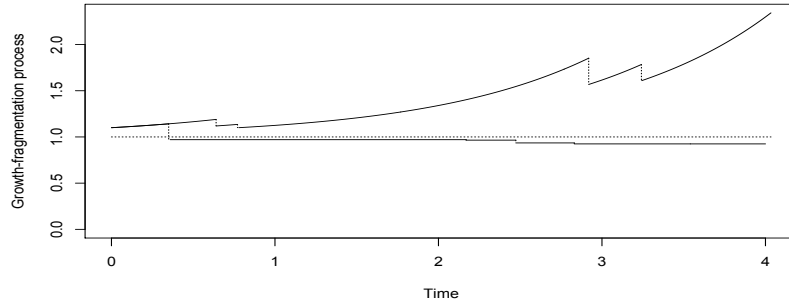


Figure 1: Two trajectories for the growth-fragmentation model. One of them is absorbed at the first jump time, while the other one seems to escape the trapping set.

This continuous-time Markov process is called absorbing because the motion may reach the absorbing interval $\Gamma = [0, 1]$ from any initial value $X_0 = x$. In this paper, we focus on the estimation of the absorption probability (see Subsection 3.2) and of the distribution of the hitting time of Γ (see Subsection 3.3) from the observation of only one trajectory of the process within a long time, that is to say from the observation of the sequences $(S_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$. We would like to emphasize that this is far to be obvious to estimate these quantities in the absorbing framework that we consider.

2.2 Semi-parametric framework

In all the sequel, we assume that we observe a PDMP $(X_t)_{t \geq 0}$ within a long time interval. In other words, we observe the n first terms of the sequence of the interarrival times $(S_k)_{k \geq 1}$ and of the sequence of the independent random loss fractions $(Y_k)_{k \geq 1}$ defined by (4).

From these independent observations, we propose to estimate the features G and λ . In the rest of the paper, we consider an estimator \hat{G}_n of the density function G and an estimator $\hat{\lambda}_n$ of the rate λ , computed from the n first loss events, that is to say from S_1, \dots, S_n and Y_1, \dots, Y_n . In some of our convergence results, we impose a few conditions on the asymptotic behaviors of \hat{G}_n and $\hat{\lambda}_n$. When one of the assumptions below will be used, this will be specified in the statement of the result.

(C₁^λ) $\hat{\lambda}_n \in [\lambda_*, \lambda^*]$, with $\lambda_* > 0$.

(C₁^G) $\|\hat{G}_n - G\|_\infty$ tends to 0 in probability.

(C₂^λ) $\hat{\lambda}_n$ tends to 0 in probability.

(C₂^G) $\int_0^1 |G(u) - \hat{G}_n(u)| u^{-1} du$ tends to 0 in probability.

In the present paper, we are not interested in the demonstration of the asymptotic properties of the estimates \hat{G}_n and $\hat{\lambda}_n$ but in the estimation of some characteristics of the PDMP $(X_t)_{t \geq 0}$ from these estimates. In particular, we establish in Theorems 3.6 and 3.8 that the convergences in probability of \hat{G}_n and $\hat{\lambda}_n$ may be transferred to our estimators of the absorption probability and of the distribution of the hitting time of Γ . Thus, we do not investigate the properties or the choice of the estimators of G and λ . Nevertheless, the assumptions that we impose on \hat{G}_n and $\hat{\lambda}_n$ are non restrictive. For the sake of readability, we introduce the following notations: \bar{S}_n denotes the empirical mean of the n first interarrival times, while the projection $\pi_{[a,b]}(x)$ is defined by

$$\pi_{[a,b]}(x) = \begin{cases} a & \text{if } x < a, \\ b & \text{if } x > b, \\ x & \text{else.} \end{cases}$$

In the case where we know two real numbers λ_* and λ^* such that $0 < \lambda_* < \lambda < \lambda^*$, the truncated maximum likelihood estimator given by

$$\hat{\lambda}_n^{ml} = \pi_{[\lambda_*, \lambda^*]}(\bar{S}_n^{-1}), \quad (5)$$

obviously satisfies both the conditions (C_{1,2}^λ). Furthermore, the Parzen-Rosenblatt estimator of a uniformly continuous density satisfies the assumption (C₁^G) whenever the bandwidth $(h_n)_{n \geq 1}$ is such that

$$\sum_{n \geq 1} \exp(-\delta n h_n^2) < \infty,$$

for any $\delta > 0$ (see [21] for instance). Finally, we will establish in Appendix C that the convergence (C₂^G) is also satisfied for the Parzen-Rosenblatt estimator under some additional conditions (see Assumption C.1) on the density of interest G .

2.3 Some applications

In this part, we present two applications of the growth-fragmentation model that we focus on in the present paper.

2.3.1 A ruin theoretical model

The growth-fragmentation process that we consider has been introduced in [16] for modeling a capital subject to random heavy loss events. We do not attempt to give an exhaustive survey about this model, but refer the reader to the paper [16] and the references therein. We consider an individual household whose income I_t at time t may be split into

$$I_t = C_t + S_t \quad (6)$$

where C_t denotes the consumption and S_t is the savings. Consumption is assumed to evolve according to

$$C_t = \begin{cases} I_t & \text{if } I_t \leq I^* \\ I^* + \alpha(I_t - I^*) & \text{else,} \end{cases} \quad (7)$$

where I^* is the critical income level and $0 < \alpha < 1$. If the income is smaller than I^* , the whole income is used for consumption. We denote by X_t the accumulated capital up to time t . The capital evolves according to

$$\frac{dX_t}{dt} = cS_t, \quad (8)$$

where $0 < c < 1$, while the income evolves with

$$I_t = bX_t, \quad (9)$$

with $b > 0$. Finally, from (6), (7), (8) and (9), the capital X_t satisfies the ordinary differential equation,

$$\frac{dX_t}{dt} = r(X_t - x^*)^+, \quad (10)$$

where $x^* = I^*/b$ and $r = (1 - a)bc$. Now, we assume that the capital X_t is subject to catastrophic events, which occur in a Poisson-like fashion with homogeneous rate λ . When an event occurs at time t , the capital X_t is reduced by a random fraction whose distribution is described by its probability density function G . After the loss event, the process starts again according to (10). We obtain a PDMP for which the interval $[0, x^*]$ is an absorbing set, called *area of poverty*. Indeed, once the process is below the critical capital x^* , the next events will reduce the capital and the process will never again reach values above x^* .

2.3.2 A Malthusian evolution

Here is an example from population dynamics. Consider a population whose the total number of individuals at time t is X_t . We assume that there exists a certain extinction threshold x^* below which the population will almost surely extinct. One of the simplest dynamic for population models is that of Malthus (see [17] for instance). In this model, a super-threshold population grows exponentially according to the equation (10). Of course, if you not enriched the model with new assumptions, starting from a super-threshold population, the population shall never be extinguished. However, note that this simple exponential growth model describes pretty well the growth of the human population over the past centuries. One way to make the model more realistic is to assume that disasters can happen. If we think at the human population, such disasters may correspond to epidemics, famines or wars. As above for the ruin theoretical model, we consider that these disasters occur in a Poisson-like fashion and that when a disaster happens, it has the effect of instantaneously reduce the population of a certain proportion. In this case, it may happen that starting from a super-threshold population, at some point in the time, a disaster affects the population so much so that it falls below the extinction threshold and therefore that extinction occurs.

3 Main results

We present here our main results on the estimation of the absorption probability and of the hitting time of Γ for the PDMP $(X_t)_{t \geq 0}$. First, we focus on a procedure for estimating the Markov kernel of the post-jump locations of this process. Almost all the proofs have been deferred into Appendix B.

3.1 Transition density of the post-jump locations

We are interested in the estimation of the transition kernel \mathcal{R} of the Markov chain of the post-jump locations $(Z_n)_{n \geq 0}$. Recall that the sequence of the jump times of the PDMP $(X_t)_{t \geq 0}$ is $(T_n)_{n \geq 0}$ and the post-jump locations are defined, for any $n \geq 0$, by $Z_n = X_{T_n}$. The Markov kernel \mathcal{R} of $(Z_n)_{n \geq 0}$ is given, for any $x \in \mathbb{R}_+^*$ and $A \in \mathcal{B}(\mathbb{R}_+^*)$, by

$$\mathcal{R}(x, A) = \mathbb{P}(Z_{n+1} \in A \mid Z_n = x).$$

In the rest of the paper, we impose the following condition on the probability density G .

Assumption 3.1 *We assume that the density G is bounded and such that $\int_0^1 G(u) u^{-1} du < \infty$.*

First, we show that \mathcal{R} may be directly computed from the characteristics G and λ .

Proposition 3.2 *The transition kernel of the Markov chain $(Z_n)_{n \geq 0}$ satisfies $\mathcal{R}(x, dy) = \mathcal{R}(x, y) dy$, with*

$$\mathcal{R}(x, y) = \begin{cases} \frac{1}{x} G\left(\frac{y}{x}\right) & \text{if } x \leq 1, \\ \frac{\lambda}{r} (x-1)^{\lambda/r} \left[\int_0^{y/x \wedge 1} G(u) u^{\lambda/r} (y-u)^{-\lambda/r-1} du \right] & \text{else.} \end{cases} \quad (11)$$

Therefore, by virtue of (11), one may propose to estimate the transition density $\mathcal{R}(x, y)$ by

$$\widehat{\mathcal{R}}_n(x, y) = \begin{cases} \frac{1}{x} \widehat{G}_n\left(\frac{y}{x}\right) & \text{if } x \leq 1, \\ \frac{\widehat{\lambda}_n}{r} (x-1)^{\widehat{\lambda}_n/r} \int_0^{y/x \wedge 1} \widehat{G}_n(u) u^{\widehat{\lambda}_n/r} (y-u)^{-\widehat{\lambda}_n/r-1} du & \text{else,} \end{cases} \quad (12)$$

where $\widehat{G}_n(\xi)$ and $\widehat{\lambda}_n$ estimate the quantities $G(\xi)$ and λ from the observation of the n first loss events. We establish that the distance between \mathcal{R} and its estimate $\widehat{\mathcal{R}}_n$ is directly related to the estimation error of \widehat{G}_n and $\widehat{\lambda}_n$.

Proposition 3.3 *Under assumption (C_1^λ) , the following inequality holds,*

$$\sup_{(x,y) \in [1,\infty) \times [0,\infty)} \left| \mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y) \right| \leq \|G - \widehat{G}_n\|_\infty + \frac{1}{\lambda_*} \left(4e^{-1} \frac{\lambda_*}{\lambda_*} + 1 \right) \|\widehat{G}_n\|_\infty |\lambda - \widehat{\lambda}_n|.$$

This result has the following corollary.

Corollary 3.4 *Under assumptions $(C_{1,2}^\lambda)$ and (C_1^G) , the estimator $\widehat{\mathcal{R}}_n$ converges towards \mathcal{R} in probability uniformly in $(x, y) \in [1, \infty) \times [0, \infty)$,*

$$\forall \varepsilon > 0, \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\sup_{(x,y) \in [1,\infty) \times [0,\infty)} \left| \mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y) \right| \geq \varepsilon \right) = 0.$$

In addition, the rate of convergence may be obtained from (24).

3.2 Absorption probability

The previous results on the estimation of the transition density \mathcal{R} allow us to estimate the absorption probability of the PDMP $(X_t)_{t \geq 0}$. When the trajectory starts from $X_0 = x > 1$, this probability is defined by

$$p(x) = \mathbb{P}(X_t \in \Gamma \text{ for some } t \mid X_0 = x),$$

where $\Gamma = [0, 1]$ is an absorbing set. We state that $p(x)$ may be found as a solution of an integral equation.

Proposition 3.5 *$p(x)$ is a solution of the following integral equation,*

$$p(x) = \int_0^1 \mathcal{R}(x, y) dy + \int_1^{+\infty} p(y) \mathcal{R}(x, y) dy. \quad (13)$$

Proof. First, we propose to rewrite $p(x)$ from the Markov chain $(Z_n)_{n \geq 0}$,

$$p(x) = \mathbb{P}(Z_n \in \Gamma \text{ for some } n \mid Z_0 = x).$$

In addition, we have

$$p(x) = \mathbb{P}(Z_1 \in \Gamma | Z_0 = x) + \mathbb{P}(Z_1 \notin \Gamma, Z_n \in \Gamma \text{ for some } n \geq 2 | Z_0 = x).$$

Together with

$$\mathbb{P}(Z_1 \notin \Gamma, Z_n \in \Gamma \text{ for some } n \geq 2 | Z_0 = x) = \int_1^\infty \mathbb{P}(Z_n \in \Gamma \text{ for some } n \geq 2 | Z_1 = y) \mathcal{R}(x, y) dy,$$

this shows (13). \square

Consequently, by virtue of (13), we propose to estimate $p(x)$ by the unique solution of the estimated integral equation

$$\hat{p}_n(x) = \int_0^1 \hat{\mathcal{R}}_n(x, y) dy + \int_1^{+\infty} \hat{p}_n(y) \hat{\mathcal{R}}_n(x, y) dy, \quad (14)$$

which satisfies both conditions

$$\lim_{x \searrow 1} \hat{p}_n(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \hat{p}_n(x) = 0.$$

Nevertheless, the above equation is not in a proper form to compute \hat{p}_n . As a consequence, we propose to solve numerically this estimated equation. On the space $L^1(1, \infty)$, endowed with its usual norm denoted by $\|\cdot\|$, we define the operator

$$\hat{K}_n : h \mapsto \int_1^{+\infty} h(y) \hat{\mathcal{R}}_n(x, y) dy, \quad (15)$$

and we introduce the following additional notation,

$$\hat{s}_n = \int_0^1 \hat{\mathcal{R}}_n(\cdot, y) dy. \quad (16)$$

Thus, the equation (14) may be rewritten as a Fredholm equation of the second kind on the space $L^1(1, \infty)$ [18],

$$\hat{p}_n - \hat{K}_n \hat{p}_n = \hat{s}_n. \quad (17)$$

As is well known, one may approximate a solution of (17) by the quantity

$$\hat{p}_{n,m} = \sum_{k=0}^m \hat{K}_n^k \hat{s}_n, \quad (18)$$

as long as $\|\hat{K}_n\| < 1$, this condition being ensured by an additional condition, as stated in the following theorem of convergence of $\hat{p}_{n,m}$ towards p .

Theorem 3.6 *Under the conditions $(C_{1,2}^\lambda)$ and (C_2^G) , and the additional assumption $\int_0^1 G(u) u^{-1} du < 1 + r/\lambda$, the equation (13) has a unique solution and moreover, $\|\hat{p}_{n,m} - p\|$ tends to 0 in probability when n and m go to infinity.*

3.3 Distribution of the hitting time

We now proceed to the estimation of $t_m(x)$, the probability for the process $(X_t)_{t \geq 0}$ starting from $X_0 = x$ to be absorbed at jump m . For $x > 1$ we have $t_1(x) = \mathbb{P}(Z_1 \in \Gamma | Z_0 = x)$ and for $m \geq 2$,

$$t_m(x) = \mathbb{P}(Z_m \in \Gamma, Z_k \notin \Gamma, 1 \leq k \leq m-1 | Z_0 = x).$$

We state in the following result that this sequence satisfies a recurrence relation.

Proposition 3.7 For $x > 1$, the functional sequence $(t_m)_{m \geq 1}$ satisfies $t_1(x) = \int_0^1 \mathcal{R}(x, y) dy$ and the recursion relation,

$$\forall m \geq 2, t_m(x) = \int_1^\infty t_{m-1}(y) \mathcal{R}(x, y) dy. \quad (19)$$

Proof. The proof follows the same reasoning as in the proof of Proposition 3.5. \square

Following the same approach as in Subsection 3.2, we propose to estimate the functional sequence $(t_m)_{m \geq 1}$ by the recursive procedure,

$$\hat{t}_{n,1}(x) = \int_0^1 \hat{\mathcal{R}}_n(x, y) dy,$$

and for $m \geq 2$,

$$\hat{t}_{n,m}(x) = \int_1^\infty \hat{t}_{n,m-1}(y) \hat{\mathcal{R}}_n(x, y) dy.$$

Using the operator \hat{K}_n defined in (15) and the notation (16), this recursion relation is closed to give

$$\hat{t}_{n,m} = \hat{K}_n^{m-1} \hat{s}_n. \quad (20)$$

Theorem 3.8 Under the conditions $(C_{1,2}^\lambda)$ and (C_2^G) , and the additional assumption $\int_0^1 G(u) u^{-1} du < 1 + r/\lambda$, then, for any integer m , $\|\hat{t}_{n,m} - t_m\|$ tends to 0 in probability when n goes to infinity.

We give the relation between the functional sequence $(\hat{t}_{n,m})_{m \geq 1}$ and the estimate $\hat{p}_{n,m}$ of the absorption probability in the following remark.

Remark 3.9 The estimation procedures for p and $(t_m)_{m \geq 1}$ may be carried out in the same time. In light of (16), (18) and (20), we have

$$\hat{p}_{n,m} = \hat{s}_n + \sum_{k=1}^m \hat{t}_{n,k}.$$

As a consequence, the estimation of the absorption probability p from the estimated sequence $(\hat{t}_{n,m})_{m \geq 1}$ does not require extra calculations.

4 Numerical illustration

This part of the paper is dedicated to some numerical illustrations of our main convergence results stated in the previous section. All the simulations have been implemented in the R language, which is commonly used in the statistical community, with an extensive use of the `integrate` function (numerical integration routine with adaptive quadrature of functions). As an example in our simulations, we choose for the probability density function G the following power function, $G(u) = 11u^{10}$ for any $u \in [0, 1]$. This density function charges the interval $[0.8, 1]$ at more than 90%. This means that the process is weakly affected by a fragmentation event. For the jump rate we choose $\lambda = 1$ and for the growth rate $r = 1$. Then,

$$\frac{\lambda}{\lambda + r} \int_0^1 G(u) u^{-1} du = 0.55 < 1,$$

so that we are in the scope of application of Theorems 3.6 and 3.8. We propose to illustrate our theoretical results Corollary 3.4 and Theorems 3.6 and 3.8 from the observation of different numbers of data ($n = 50, 75$ and 100). In addition, we always present the distribution of our estimates from a fixed number of data over 100 replicates of the numerical experiment.

For these simulation experiments, we choose to estimate the density $G(x)$ by the Parzen-Rosenblatt estimator $\hat{G}_n^{\text{PR}}(x)$ defined by

$$\hat{G}_n^{\text{PR}}(x) = \frac{1}{nh_n} \sum_{i=1}^n \mathbb{K}\left(\frac{Y_i - x}{h_n}\right),$$

where \mathbb{K} is the Gaussian kernel and the parameter h_n is the bandwidth. The estimator is computed from the R function `density` with an optimal choice of the bandwidth parameter. In addition, λ is estimated from the observations S_i 's by the truncated maximum likelihood estimator $\hat{\lambda}_n^{\text{ml}}$ defined in (5). These estimates satisfy the conditions that we impose in the paper.

First, we present some simulation results for the transition kernel $\mathcal{R}(x, y)$. The transition kernel is not really a quantity of interest in the model in contrary to the rate and measure of jumps λ and $G(u)du$. Nevertheless, the kernel appears when we want to compute the probability of hitting, or the hitting time of, Γ . This is therefore required to be able to estimate $\mathcal{R}(x, y)$ in our approach. Recall the definition (12) of the estimator of \mathcal{R} from $\hat{\lambda}_n$ and \hat{G}_n^{PR} . In Figure 2 are displayed the pointwise error with boxplots over 100 replications between $\mathcal{R}(x, 2)$ and its estimate for $1 \leq x \leq 5$ and between $\mathcal{R}(2, y)$ and its estimate for $1 \leq y \leq 5$ for $n = 50, 75$ and 100 data. The corresponding integrated square errors are given in Figure 3. In both cases, we observe a decrease in the error when the number of data grows, despite the low number of data. However, this is not very surprising here since the transition kernel is estimated from its exact expression (see Proposition 3.2), substituting λ by $\hat{\lambda}_n$ and G by \hat{G}_n^{PR} .

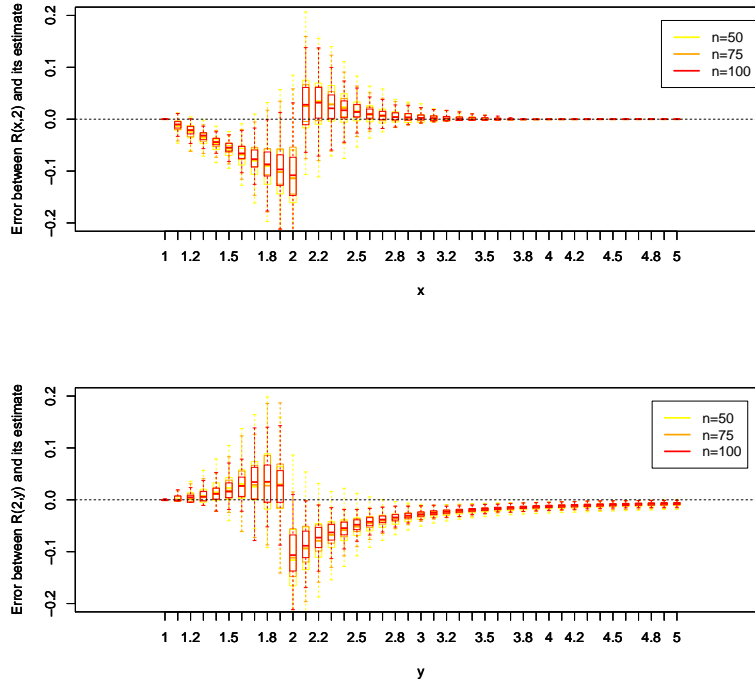


Figure 2: Pointwise error on 100 replicates between $\mathcal{R}(x, 2)$ and its estimate for $1 \leq x \leq 5$ (top), and between $\mathcal{R}(2, y)$ and its estimate for $1 \leq y \leq 5$ (bottom), from the observation of $n = 50, 75$ or 100 random loss events. The corresponding integrated square errors are given in Figure 3.

Now, we proceed to the simulation of the estimation of $p(x)$, the probability for the process $(X_t)_{t \geq 0}$ to be absorbed by $\Gamma = [0, 1]$ starting from $x > 1$. This is, with the time of absorption, one of the two main quantities of interest in the model. Indeed, for the ruin theoretical model of Section 2.3.1, $p(x)$ corresponds

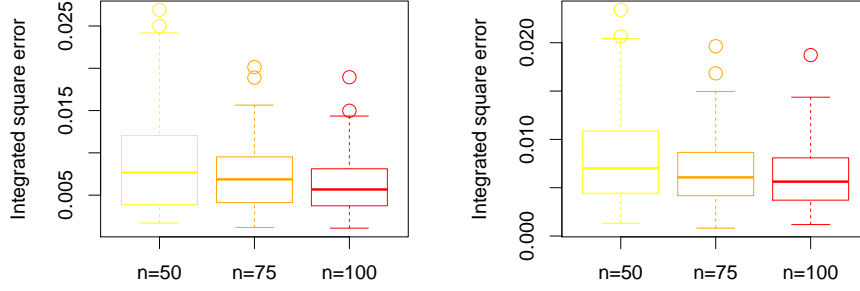


Figure 3: Integrated square error on 100 replicates between $\mathcal{R}(\cdot, 2)$ and its estimate (left), and between $\mathcal{R}(2, \cdot)$ and its estimate (right), from the observation of $n = 50, 75$ or 100 random loss events. The corresponding pointwise errors are given in Figure 2.

to the probability to be ruin starting from some capital x . For the Malthusian evolution of Section 2.3.2, $p(x)$ is the probability for a population of initial size x to extinct. Nevertheless, we can not compute directly the function of interest p . As a consequence, we propose to compare $\hat{p}_{n,m}$ and the numerical approximation $p_m = \sum_{k=0}^m K^k s$ of p , where the operator K is defined in (25) and $s = \int_0^1 \mathcal{R}(\cdot, y) dy$. Roughly speaking, K and s are the deterministic limits of the estimates \hat{K}_n and \hat{s}_n presented in (15) and (16). The error in $L^1(1, \infty)$ -norm between p and p_m satisfies

$$\|p - p_m\| \leq \|s\| \frac{\|K\|^{m+1}}{1 - \|K\|}.$$

Together with the chosen numerical values and $m = 10$, we have $\|p - p_m\| \leq 1.6 \times 10^{-4}$. Consequently, the numerical error due to the approximation of p is very low and does not affect our comparison results presented in the sequel.

Recall that our approximation of p is given by $\hat{p}_{n,m} = \sum_{k=0}^m \hat{K}_n^k \hat{s}_n$. In the simulations, we compare $\hat{p}_{n,m}$ with p_m for $m = 10$ and $n = 50, 70$ and 100 data. Figure 4 displays the shape of p_m and $\hat{p}_{n,m}$ as well as the boxplots of the punctual errors between the two curves. The corresponding integrated square error is presented in Figure 5. In both cases, a decrease in the error is observed when n grows. Note that the error is already small for $n = 50$ and seems to behave quite well despite the successive application of the kernel \hat{K}_n .

Finally, we go on with the estimation of $t_m(x)$, the probability for the process $(X_t)_{t \geq 0}$ starting from x , to be absorbed at jump m . The quantity $t_m(x)$ is an important feature of the model and provides additional information to that given by $p(x)$. Remark that according to Proposition 3.7, t_m may be computed in an exact way on the contrary of $p(x)$. There is therefore no numerical error in this case (if we do not consider the numerical errors introduced by the computation of the kernel integrals). Thus, we compare directly $t_m(x)$ with its estimator $\hat{t}_{n,m}(x)$ given by equation (20). At first, we notice that the estimation of the probability of absorption $\hat{p}_{n,m}$ and the estimation of the times at which an absorption occurs $\hat{t}_{n,m}(x)$ are related through the formula,

$$\hat{p}_{n,m} = \sum_{k=0}^m \hat{t}_{n,k+1}.$$

Therefore, in our previous computations of $\hat{p}_{n,m}$, we already have computed the quantities $\hat{t}_{n,m}$ and no

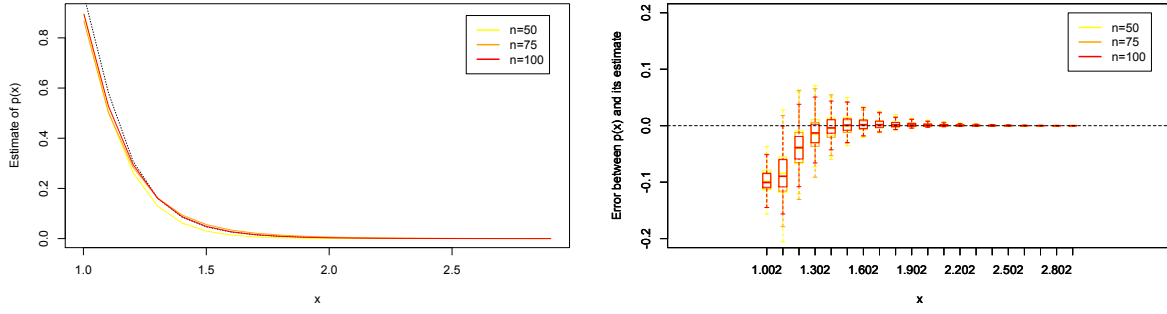


Figure 4: The absorption probability p (approximated by p_m) and its estimates $\hat{p}_{n,m}$ from the observation of $n = 50, 75$ or 100 random loss events and for $m = 10$ iterations of the estimated kernel \hat{K}_n (top), and associated pointwise error on 100 replicates (bottom). The corresponding integrated square error is given in Figure 5.

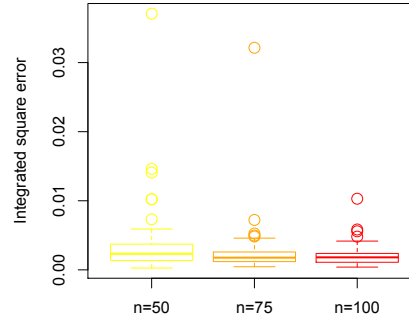


Figure 5: Integrated square error on 100 replicates between the absorption probability p (approximated by p_m) and its estimates $\hat{p}_{n,m}$ from the observation of $n = 50, 75$ or 100 random loss events and for $m = 10$ iterations of the estimated kernel \hat{K}_n . The corresponding pointwise error is given in Figure 4 (bottom).

further calculations are required. In Figure 6, we present the integrated square error between t_m and its estimate $\hat{t}_{n,m}$ from the observation of $n = 50, 75$ or 100 random loss events and for $m = 1, 2, 3$ and 4 , that is for the four first absorption times. There is a decrease of the error when n grows for each value of m . Quantitatively, this does not make sense to compare the error for $m = 2$ and $m = 4$ since, as displayed in Figure 7, the order of magnitude of the estimated probabilities is not at all the same. Figure 7 presents the distribution of the hitting time of Γ , $t_m(x)$, for $x = 1.1$ and $m = 1, \dots, 10$, and also the distribution of its estimates $\hat{t}_{n,m}$ from the observation of $n = 50, 75$ or 100 random loss events. More precisely, in this figure is represented the mean of the estimators together with the first and third quartiles, over 100 replications. Once again, a decrease in the error was observed when n grows showing that the law of the hitting times of Γ is well estimated. These results, coupled with the estimate of $p(x)$, give all the interesting information in the study of this model. In all the procedure, the estimates are of high quality despite the low number of data used, in particular with respect to the sample sizes used in the works [1, 2, 12, 13].

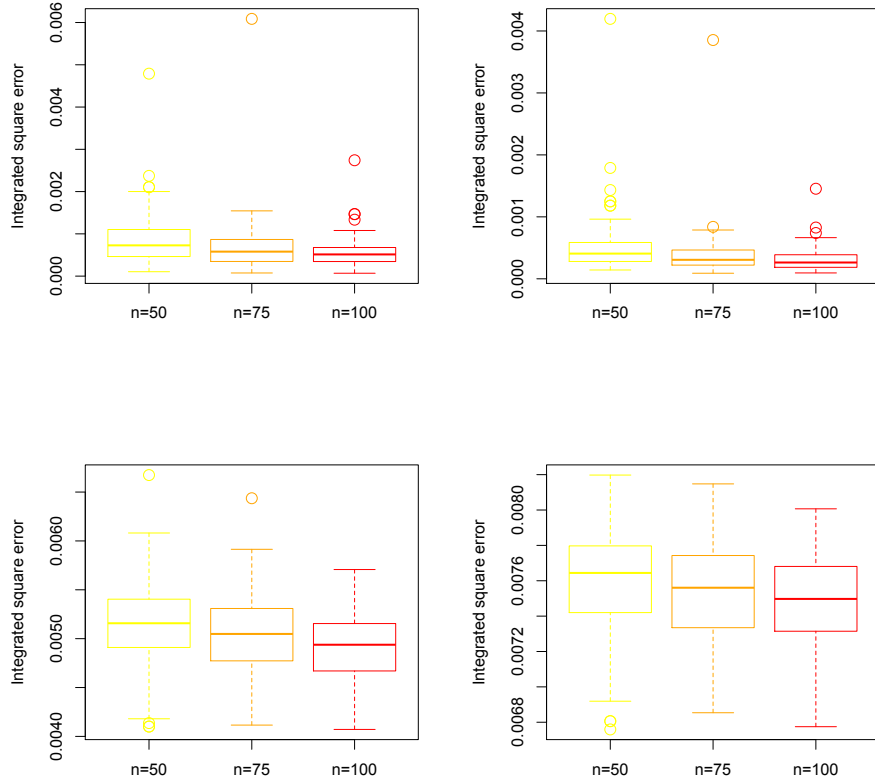


Figure 6: Integrated square error on 100 replicates between t_m and its estimate $\hat{t}_{n,m}$ from the observation of $n = 50, 75$ or 100 random loss events and for $m = 1$ (top left), $m = 2$ (top right), $m = 3$ (bottom left) and $m = 4$ (bottom right).

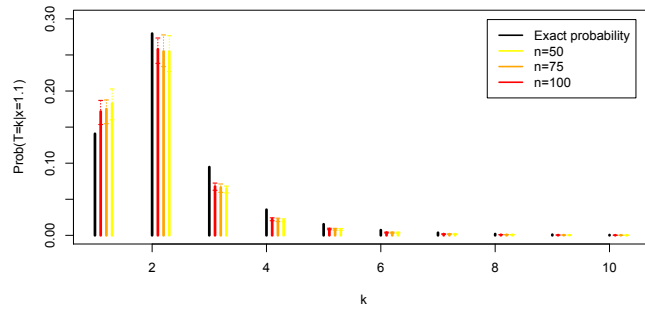


Figure 7: Distribution of the hitting time of $\Gamma t_m(x)$ for $x = 1.1$ and $m = 1, \dots, 10$ and its estimates $\hat{t}_{n,m}$ from the observation of $n = 50, 75$ or 100 random loss events. The integrated square error between t_m and $\hat{t}_{n,m}$ is given in Figure 6.

A Some technical lemmas

This part is dedicated to the presentation of some technical results which will be useful in the proofs of our main results presented in Appendix B. For convenience, we use in the sequel the following notations. For $\lambda > 0$, $x \geq 1$ and $y \geq 0$, we define

$$\alpha_\lambda(x) = \frac{(x-1)^{\lambda/r}}{r}, \quad \beta_\lambda(y, u) = u^{\lambda/r} (y-u)^{-\lambda/r-1} \quad \text{and} \quad f_\lambda(x, y) = \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) du. \quad (21)$$

Lemma A.1 *For any $x > 1$, $y \geq x$, the deterministic flow (2) satisfies $\Phi_x(t) = y$ if and only if $t = \frac{1}{r} \log \left(\frac{y-1}{x-1} \right)$.*

Proof. This result is obvious. \square

Lemma A.2 *For any $\lambda_1, \lambda_2 \in [\lambda_*, \lambda^*]$, $x \geq 1$, $y \geq 0$ and $u \in (0, y/x)$, we have*

$$\begin{aligned} \text{(i)} \quad & f_\lambda(x, y) \leq \frac{1}{\lambda} \mathbb{1}_{\{y < x\}} \left[1 - \left(\frac{x-1}{x} \right)^{\lambda/r} \right] + \frac{1}{\lambda} \mathbb{1}_{\{y \geq x\}} \left[\left(\frac{x-1}{y-1} \right)^{\lambda/r} - \left(\frac{x-1}{y} \right)^{\lambda/r} \right] \leq \frac{1}{\lambda}, \\ \text{(ii)} \quad & \left| \alpha_{\lambda_1}(x) \beta_{\lambda_1}(y, u) - \alpha_{\lambda_2}(x) \beta_{\lambda_2}(y, u) \right| \leq \frac{1}{r^2} \frac{1}{y-u} \left(\frac{u(x-1)}{y-u} \right)^{\lambda_*/r} \log \left(\frac{u(x-1)}{y-u} \right) |\lambda_1 - \lambda_2|, \\ \text{(iii)} \quad & \sup_{x \geq 1, y \geq 0} \int_0^{y/x \wedge 1} \left| \alpha_{\lambda_1}(x) \beta_{\lambda_1}(y, u) - \alpha_{\lambda_2}(x) \beta_{\lambda_2}(y, u) \right| du \leq \frac{4e^{-1}}{\lambda_*^2} |\lambda_1 - \lambda_2|. \end{aligned}$$

Proof. We begin with the inequality (i). Let (x, y) be in $[1, \infty) \times [0, \infty)$. For any $u \in [0, y/x \wedge 1]$ we have

$$0 \leq u^{\lambda/r} (y-u)^{-\lambda/r-1} \leq (y/x \wedge 1)^{\lambda/r} (y-u)^{-\lambda/r-1}.$$

Therefore,

$$f_\lambda(x, y) \leq \frac{1}{r} (y/x \wedge 1)^{\lambda/r} (x-1)^{\lambda/r} \int_0^{y/x \wedge 1} (y-u)^{-\lambda/r-1} du.$$

Computing the integral leads to

$$f_\lambda(x, y) \leq \frac{1}{\lambda} (y/x \wedge 1)^{\lambda/r} (x-1)^{\lambda/r} \left[(y - y/x \wedge 1)^{-\lambda/r} - y^{-\lambda/r} \right].$$

We now split the latter term in two using the elementary fact that $1 = \mathbb{1}_{\{y < x\}} + \mathbb{1}_{\{y \geq x\}}$. This yields

$$\begin{aligned} f_\lambda(x, y) & \leq \frac{1}{\lambda} (y/x)^{\lambda/r} (x-1)^{\lambda/r} \mathbb{1}_{\{y < x\}} \left[(y - y/x)^{-\lambda/r} - y^{-\lambda/r} \right] \\ & \quad + \frac{1}{\lambda} (x-1)^{\lambda/r} \mathbb{1}_{\{y \geq x\}} \left[(y-1)^{-\lambda/r} - y^{-\lambda/r} \right] \\ & = \frac{1}{\lambda} \mathbb{1}_{\{y < x\}} \left[1 - \left(\frac{x-1}{x} \right)^{\lambda/r} \right] \\ & \quad + \frac{1}{\lambda} \mathbb{1}_{\{y \geq x\}} \left[\left(\frac{x-1}{y-1} \right)^{\lambda/r} - \left(\frac{x-1}{y} \right)^{\lambda/r} \right]. \end{aligned}$$

Using the fact that when $y \geq x$, $\frac{x-1}{y-1} \leq 1$ and noticing that the two terms $\frac{x-1}{y}$ and $\frac{x-1}{x}$ are non negative, we obtain:

$$f_\lambda(x, y) \leq \frac{1}{\lambda} \mathbb{1}_{\{y < x\}} + \frac{1}{\lambda} \mathbb{1}_{\{y \geq x\}} = \frac{1}{\lambda}.$$

We go on with the second part of the lemma. One may derivate with respect to λ to obtain

$$\partial_\lambda \alpha_\lambda(x) \beta_\lambda(y, u) = \frac{1}{r^2} \frac{1}{y-u} \left(\frac{u(x-1)}{y-u} \right)^{\lambda/r} \log \left(\frac{u(x-1)}{y-u} \right).$$

Then, we use that for $x \geq 1$, $y \geq 0$ and $u \in [0, y/x \wedge 1]$, one has $X = \frac{u(x-1)}{y-u} \in (0, 1]$ such that $X^{\lambda/2r} |\log X| \leq \frac{2r}{\lambda} e^{-1}$. This fact yields

$$|\partial_\lambda \alpha_\lambda(x) \beta_\lambda(y, u)| \leq \frac{2e^{-1}}{r\lambda} \frac{1}{y-u} \left(\frac{u(x-1)}{y-u} \right)^{\lambda/2r} \leq \frac{2e^{-1}}{r\lambda_*} \frac{1}{y-u} \left(\frac{u(x-1)}{y-u} \right)^{\lambda_*/2r}.$$

Notice that the last inequality is uniform in $\lambda \in [\lambda_*, \lambda^*]$. This proves the second assertion (ii). For the third one, using the mean value theorem and similar calculations as above, we obtain that

$$\begin{aligned} \int_0^{y/x \wedge 1} |\alpha_{\lambda_1}(x) \beta_{\lambda_1}(y, u) - \alpha_{\lambda_2}(x) \beta_{\lambda_2}(y, u)| du &\leq \mathbb{1}_{\{y < x\}} \frac{4e^{-1}}{\lambda_*^2} \left[1 - \left(\frac{x-1}{x} \right)^{\lambda_*/2r} \right] |\lambda_1 - \lambda_2| \\ &\quad + \mathbb{1}_{\{y \geq x\}} \frac{4e^{-1}}{\lambda_*^2} \left[\left(\frac{x-1}{y-1} \right)^{\lambda_*/2r} - \left(\frac{x-1}{y} \right)^{\lambda_*/2r} \right] |\lambda_1 - \lambda_2|. \end{aligned}$$

Using the fact that when $y \geq x$, $\frac{x-1}{y-1} \leq 1$ and noticing that the two terms $\frac{x-1}{y}$ and $\frac{x-1}{x}$ are non negative, we obtain

$$\begin{aligned} \int_0^{y/x \wedge 1} |\alpha_{\lambda_1}(x) \beta_{\lambda_1}(y, u) - \alpha_{\lambda_2}(x) \beta_{\lambda_2}(y, u)| du &\leq \mathbb{1}_{\{y < x\}} \frac{4e^{-1}}{\lambda_*^2} |\lambda_1 - \lambda_2| + \mathbb{1}_{\{y \geq x\}} \frac{4e^{-1}}{\lambda_*^2} |\lambda_1 - \lambda_2| \\ &= \frac{4e^{-1}}{\lambda_*^2} |\lambda_1 - \lambda_2|. \end{aligned}$$

The result follows. □

Lemma A.3 *The following equality holds,*

$$\sup_{y \in [0, \infty[} \int_1^\infty \mathcal{R}(x, y) dx = \frac{\lambda}{\lambda + r} \int_0^1 \frac{G(u)}{u} du.$$

Proof. By definition of \mathcal{R} one may write

$$\int_1^\infty \mathcal{R}(x, y) dx = \lambda \int_1^\infty \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) G(u) du dx.$$

In the above term, one may change the order of integration to integrate in x at first. We obtain

$$\begin{aligned} \lambda \int_1^\infty \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) G(u) du dx &= \lambda \int_0^1 \int_1^{y/u} \alpha_\lambda(x) dx \beta_\lambda(y, u) G(u) \mathbb{1}_{\{u \leq y\}} du \\ &= \frac{\lambda}{r} \int_0^1 \frac{1}{\lambda/r + 1} \left(\frac{y}{u} - 1 \right)^{\lambda/r + 1} \beta_\lambda(y, u) G(u) \mathbb{1}_{\{u \leq y\}} du \\ &= \frac{\lambda}{\lambda + r} \int_0^1 G(u) \mathbb{1}_{\{u \leq y\}} \frac{du}{u}. \end{aligned}$$

The result follows. □

Lemma A.4 Under Assumption (C_1^λ) , almost-surely the following inequality holds

$$\sup_{y \in [0, \infty)} \int_1^\infty |\mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y)| dx \leq \frac{\lambda}{\lambda + r} \int_0^1 |G(u) - \widehat{G}_n(u)| \frac{du}{u} + \lambda^* \left[4e^{-1} \frac{1}{\lambda_*^2} + \frac{1}{\lambda_* + r} \right] \int_0^1 |\widehat{G}_n(u)| \frac{du}{u} |\lambda - \widehat{\lambda}_n|.$$

Proof. One may write

$$\begin{aligned} \mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y) &= \lambda \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) [G(u) - \widehat{G}_n(u)] du \\ &\quad + \lambda \int_0^{y/x \wedge 1} \widehat{G}_n(u) [\alpha_\lambda(x) \beta_\lambda(y, u) - \alpha_{\widehat{\lambda}_n}(x) \beta_{\widehat{\lambda}_n}(y, u)] du \\ &\quad + (\lambda - \widehat{\lambda}_n) \alpha_{\widehat{\lambda}_n}(x) \int_0^{y/x \wedge 1} \beta_{\widehat{\lambda}_n}(y, u) \widehat{G}_n(u) du. \end{aligned}$$

Thus, for any $y \geq 0$, we have

$$\begin{aligned} \int_1^\infty |\mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y)| dx &\leq \lambda \int_1^\infty \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) |G(u) - \widehat{G}_n(u)| du dx \\ &\quad + \lambda \int_1^\infty \int_0^{y/x \wedge 1} |\widehat{G}_n(u)| |\alpha_\lambda(x) \beta_\lambda(y, u) - \alpha_{\widehat{\lambda}_n}(x) \beta_{\widehat{\lambda}_n}(y, u)| du dx \\ &\quad + |\lambda - \widehat{\lambda}_n| \int_1^\infty \alpha_{\widehat{\lambda}_n}(x) \int_0^{y/x \wedge 1} \beta_{\widehat{\lambda}_n}(y, u) |\widehat{G}_n(u)| du dx. \end{aligned}$$

In the three above terms, one may change the order of integration to integrate in x at first. Let us deal with these terms separately. For the first term, we have

$$\begin{aligned} \lambda \int_1^\infty \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) |G(u) - \widehat{G}_n(u)| du dx &= \lambda \int_0^1 \int_1^{y/u} \alpha_\lambda(x) dx \beta_\lambda(y, u) |G(u) - \widehat{G}_n(u)| \mathbb{1}_{\{u \leq y\}} du \\ &= \frac{\lambda}{r} \int_0^1 \frac{1}{\lambda/r + 1} \left(\frac{y}{u} - 1 \right)^{\lambda/r + 1} \beta_\lambda(y, u) |G(u) - \widehat{G}_n(u)| \mathbb{1}_{\{u \leq y\}} du \\ &= \frac{\lambda}{\lambda + r} \int_0^1 |G(u) - \widehat{G}_n(u)| \mathbb{1}_{\{u \leq y\}} \frac{du}{u}. \end{aligned}$$

For the third term, a similar calculation gives

$$|\lambda - \widehat{\lambda}_n| \int_1^\infty \alpha_{\widehat{\lambda}_n}(x) \int_0^{y/x \wedge 1} \beta_{\widehat{\lambda}_n}(y, u) |\widehat{G}_n(u)| du dx = |\lambda - \widehat{\lambda}_n| \frac{\widehat{\lambda}_n}{\widehat{\lambda}_n + r} \int_0^1 |\widehat{G}_n(u)| \mathbb{1}_{\{u \leq y\}} \frac{du}{u}.$$

The most intricate term is the second. Using Lemma A.2, we have

$$\begin{aligned} \lambda \int_1^\infty \int_0^{y/x \wedge 1} \widehat{G}_n(u) |\alpha_\lambda(x) \beta_\lambda(y, u) - \alpha_{\widehat{\lambda}_n}(x) \beta_{\widehat{\lambda}_n}(y, u)| du dx &\leq \lambda |\lambda - \widehat{\lambda}_n| \int_1^\infty \int_0^{y/x \wedge 1} \frac{2e^{-1}}{r\lambda_*} \frac{1}{y - u} \left(\frac{u(x-1)}{y-u} \right)^{\lambda_*/2r} |\widehat{G}_n(u)| du dx \\ &= \frac{\lambda 2e^{-1}}{r\lambda_*} |\lambda - \widehat{\lambda}_n| \int_0^1 \frac{1}{y-u} \left(\frac{u}{y-u} \right)^{\lambda_*/2r} |\widehat{G}_n(u)| \int_1^{y/u} (x-1)^{\lambda_*/2r} dx \mathbb{1}_{\{u \leq y\}} du \\ &= \frac{4\lambda^* e^{-1}}{\lambda_*^2} |\lambda - \widehat{\lambda}_n| \int_0^1 |\widehat{G}_n(u)| \mathbb{1}_{\{u \leq y\}} \frac{du}{u}. \end{aligned}$$

The result follows by aggregation of the three above estimates. \square

B Proofs of the main results

This section gathers the proofs of the different propositions stated in Section 3.

B.1 Proof of Proposition 3.2

In both cases $x > 1$ and $x \leq 1$, we have

$$\mathcal{R}(x, dy) = \left[\int_{\mathbb{R}_+} \frac{1}{z} G\left(\frac{y}{z}\right) \mathcal{S}(x, dz) \right] dy, \quad (22)$$

where the conditional distribution $\mathcal{S}(x, dz)$ is defined from its cumulative version,

$$\mathcal{S}(x, (-\infty, z]) = \mathbb{P}(\Phi_{Z_{n-1}}(S_n) \leq z \mid Z_{n-1} = x).$$

For $x \leq 1$, from (2), we have $\mathcal{S}(x, dz) = \delta_{\{x\}}(dz)$. This shows (11) for $x \leq 1$. If $x > 1$, for any $z \geq x$,

$$\mathcal{S}(x, (-\infty, z]) = \mathbb{P}\left(S_n \leq \frac{1}{r} \log\left(\frac{z-1}{x-1}\right)\right),$$

according to Lemma A.1. As a consequence,

$$\mathcal{S}(x, dz) = \frac{\lambda}{r} \frac{(x-1)^{\lambda/r}}{(z-1)^{\lambda/r+1}} \mathbb{1}_{[x, \infty)}(z) dz. \quad (23)$$

Since G is a probability density function on $[0, 1]$, together with (23), we may re-write (22) as

$$\mathcal{R}(x, dy) = \frac{\lambda}{r} (x-1)^{\lambda/r} \left[\int_{x \vee y}^{+\infty} G\left(\frac{y}{z}\right) \frac{(z-1)^{-\lambda/r-1}}{z} dz \right] dy$$

By the change of variable $u = y/z$, we obtain

$$\mathcal{R}(x, y) = \frac{\lambda}{r} (x-1)^{\lambda/r} \left[\int_0^{y/x \wedge 1} G(u) \left(\frac{y-u}{u}\right)^{-\lambda/r-1} u^{-1} du \right].$$

This shows the result (11) for $x > 1$.

B.2 Proof of Proposition 3.3

Let $n \in \mathbb{N}$, $\lambda \in [\lambda_*, \lambda^*]$, $x \geq 1$ and $y \geq 0$. We work ω by ω so that the desired almost-sure inequality will follow. Recall that by equations (11) and (12) together with the notations (21),

$$\mathcal{R}(x, y) = \lambda \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) G(u) du$$

and

$$\widehat{\mathcal{R}}_n(x, y) = \widehat{\lambda}_n \alpha_{\widehat{\lambda}_n} \int_0^{y/x \wedge 1} \beta_{\widehat{\lambda}_n}(y, u) \widehat{G}_n(u) du.$$

By an elementary rearranging, one may write

$$\begin{aligned}\mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y) &= \lambda \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) \left[G(u) - \widehat{G}_n(u) \right] du \\ &\quad + \lambda \int_0^{y/x \wedge 1} \widehat{G}_n(u) \left[\alpha_\lambda(x) \beta_\lambda(y, u) - \alpha_{\widehat{\lambda}_n}(x) \beta_{\widehat{\lambda}_n}(y, u) \right] du \\ &\quad + \left(\lambda - \widehat{\lambda}_n \right) \alpha_{\widehat{\lambda}_n}(x) \int_0^{y/x \wedge 1} \beta_{\widehat{\lambda}_n}(y, u) \widehat{G}_n(u) du.\end{aligned}$$

We deal with the three above terms separately. For the first term we have

$$\lambda \alpha_\lambda(x) \left| \int_0^{y/x \wedge 1} \beta_\lambda(y, u) \left[G(u) - \widehat{G}_n(u) \right] du \right| \leq \lambda \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) du \|G - \widehat{G}_n\|_\infty.$$

Thus, by the first part of Lemma A.2,

$$\left| \lambda \alpha_\lambda(x) \int_0^{y/x \wedge 1} \beta_\lambda(y, u) \left[G(u) - \widehat{G}_n(u) \right] du \right| \leq \lambda f_\lambda(x, y) \|G - \widehat{G}_n\|_\infty \leq \|G - \widehat{G}_n\|_\infty.$$

Now for the second term, using this time the third part of Lemma A.2,

$$\begin{aligned}\lambda \left| \int_0^{y/x \wedge 1} \widehat{G}_n(u) \left[\alpha_\lambda(x) \beta_\lambda(y, u) - \alpha_{\widehat{\lambda}_n}(x) \beta_{\widehat{\lambda}_n}(y, u) \right] du \right| \\ \leq \lambda \|\widehat{G}_n(u)\|_\infty \int_0^{y/x \wedge 1} \left| \alpha_\lambda(x) \beta_\lambda(y, u) - \alpha_{\widehat{\lambda}_n}(x) \beta_{\widehat{\lambda}_n}(y, u) \right| du \\ \leq \lambda \|\widehat{G}_n(u)\|_\infty \frac{4e^{-1}}{\lambda_*^2} |\lambda - \widehat{\lambda}_n| \\ \leq \frac{4\lambda^* e^{-1}}{\lambda_*^2} \|\widehat{G}_n(u)\|_\infty |\lambda - \widehat{\lambda}_n|.\end{aligned}$$

For the last term we have, using again the first part of Lemma A.2,

$$|\lambda - \widehat{\lambda}_n| \alpha_{\widehat{\lambda}_n}(x) \int_0^{y/x \wedge 1} \beta_{\widehat{\lambda}_n}(y, u) \widehat{G}_n(u) du \leq |\lambda - \widehat{\lambda}_n| \|\widehat{G}_n\|_\infty f_{\widehat{\lambda}_n}(x, y) \leq |\lambda - \widehat{\lambda}_n| \frac{\|\widehat{G}_n\|_\infty}{\lambda_*}.$$

This ends the proof.

B.3 Proof of Corollary 3.4

Let us introduce the notations

$$I_n = \sup_{(x, y) \in [1, \infty) \times [0, +\infty)} \left| \mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y) \right| \quad \text{and} \quad C = \frac{1}{\lambda_*} \left(4e^{-1} \frac{\lambda^*}{\lambda_*} + 1 \right).$$

For any $\varepsilon > 0$, according to Proposition 3.3, we have

$$\begin{aligned}\mathbb{P}(I_n \geq \varepsilon) &\leq \mathbb{P} \left(\|G - \widehat{G}_n\|_\infty + C \|\widehat{G}_n\|_\infty |\lambda - \widehat{\lambda}_n| \geq \varepsilon \right) \\ &\leq \mathbb{P} \left(\|G - \widehat{G}_n\|_\infty \geq \frac{\varepsilon}{2} \right) + \mathbb{P} \left(C \|\widehat{G}_n\|_\infty |\lambda - \widehat{\lambda}_n| \geq \frac{\varepsilon}{2} \right).\end{aligned}$$

Let η be a positive real. Using the elementary inequality satisfied for any reals a and b ,

$$ab \leq \frac{1}{4\eta}a^2 + \eta b^2,$$

we have

$$\begin{aligned} \mathbb{P}\left(\|\widehat{G}_n\|_\infty \left|\lambda - \widehat{\lambda}_n\right| \geq \frac{\varepsilon}{2C}\right) &\leq \mathbb{P}\left(\eta\|\widehat{G}_n\|_\infty^2 + \frac{1}{4\eta}\left|\lambda - \widehat{\lambda}_n\right|^2 \geq \frac{\varepsilon}{2C}\right) \\ &\leq \mathbb{P}\left(\eta\|\widehat{G}_n\|_\infty^2 \geq \frac{\varepsilon}{4C}\right) + \mathbb{P}\left(\frac{1}{4\eta}\left|\lambda - \widehat{\lambda}_n\right|^2 \geq \frac{\varepsilon}{4C}\right). \end{aligned}$$

Notice that

$$\begin{aligned} \mathbb{P}\left(\|\widehat{G}_n\|_\infty^2 \geq \frac{\varepsilon}{4C\eta}\right) &= \mathbb{P}\left(\|\widehat{G}_n\|_\infty \geq \sqrt{\frac{\varepsilon}{4C\eta}}\right) \\ &\leq \mathbb{P}\left(\|G\|_\infty + \|\widehat{G}_n - G\|_\infty \geq \sqrt{\frac{\varepsilon}{4C\eta}}\right) \\ &\leq \mathbb{P}\left(\|G\|_\infty \geq \frac{1}{2}\sqrt{\frac{\varepsilon}{4C\eta}}\right) + \mathbb{P}\left(\|\widehat{G}_n - G\|_\infty \geq \frac{1}{2}\sqrt{\frac{\varepsilon}{4C\eta}}\right). \end{aligned}$$

With $\eta^* = \frac{\varepsilon}{16C(\|G\|_\infty+1)^2}$, we have

$$\mathbb{P}\left(\|G\|_\infty \geq \frac{1}{2}\sqrt{\frac{\varepsilon}{4C\eta}}\right) = 0.$$

Thus,

$$\mathbb{P}\left(\|\widehat{G}_n\|_\infty^2 \geq \frac{\varepsilon}{4C\eta^*}\right) \leq \mathbb{P}\left(\|\widehat{G}_n - G\|_\infty \geq \frac{1}{2}\sqrt{\frac{\varepsilon}{4C\eta^*}}\right).$$

To sum up,

$$\mathbb{P}(I_n \geq \varepsilon) \leq \mathbb{P}\left(\|\widehat{G}_n - G\|_\infty \geq \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\|\widehat{G}_n - G\|_\infty \geq \|G\|_\infty + 1\right) + \mathbb{P}\left(\left|\lambda - \widehat{\lambda}_n\right| \geq \frac{\varepsilon}{4C(1 + \|G\|_\infty)}\right). \quad (24)$$

The result follows.

B.4 Proof of Theorem 3.6

Let us define the operator

$$K : h \mapsto \int_1^{+\infty} h(y)\mathcal{R}(x,y)dy \quad (25)$$

on $L^1(1, \infty)$. Let us show that the norm of the operator K on $L^1(1, \infty)$ is less than 1 if the condition $\frac{\lambda}{\lambda+r} \int_0^1 \frac{G(u)}{u} du < 1$ is satisfied. Indeed, using Jensen and Fubini's theorems, for any $h \in L^1(1, \infty)$ we have,

$$\|Kh\| = \int_1^\infty \left| \int_1^\infty h(y)\mathcal{R}(x,y) dy \right| dx \leq \int_1^\infty \int_1^\infty |h(y)|\mathcal{R}(x,y) dy dx \leq \sup_{y \in [1, \infty[} \int_1^\infty \mathcal{R}(x,y) dx \int_1^\infty |h(y)| dy.$$

According to Lemma A.3, the above inequalities yield

$$\|Kh\| \leq \frac{\lambda}{\lambda+r} \int_0^1 \frac{G(u)}{u} du \|h\|.$$

Therefore, under the condition, $\frac{\lambda}{\lambda+r} \int_0^1 \frac{G(u)}{u} du < 1$, we get $\|K\| < 1$. One may then rewrite equation (13) as the Fredholm equation

$$p - Kp = s,$$

where $s(x) = \int_0^1 \mathcal{R}(x, y) dy$. This equation has obviously a unique solution since $\|K\| < 1$. Notice that $\|p\| < \infty$ since one may write $p = \sum_{k=0}^{\infty} K^k s$. The following proposition precises the relations between \widehat{K}_n with K and \widehat{s}_n with s .

Proposition B.1 *The estimation \widehat{K}_n and \widehat{s}_n converge toward K and s respectively in probability. For any $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\|\widehat{K}_n - K\| \geq \varepsilon \right) = 0, \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\|\widehat{s}_n - s\| \geq \varepsilon \right) = 0.$$

Proof. First, let us notice that for any $h \in L^1(1, \infty)$, we have

$$\begin{aligned} \|(K - \widehat{K}_n)h\| &= \int_1^\infty \left| \int_1^\infty h(y) (\mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y)) dy \right| dx \\ &\leq \int_1^\infty \int_1^\infty |h(y)| |\mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y)| dy dx \\ &\leq \|h\| \sup_{y \geq 1} \int_1^\infty |\mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y)| dx. \end{aligned}$$

Therefore,

$$\|K - \widehat{K}_n\| \leq \sup_{y \geq 1} \int_1^\infty |\mathcal{R}(x, y) - \widehat{\mathcal{R}}_n(x, y)| dx$$

\mathbb{P} -a.s. Then, using Lemma A.4, almost-surely we have

$$\|K - \widehat{K}_n\| \leq \frac{\lambda}{\lambda+r} \int_0^1 |G(u) - \widehat{G}_n(u)| u^{-1} du + \lambda^* \left(4e^{-1} \frac{1}{\lambda_*^2} + \frac{1}{\lambda_* + r} \right) \int_0^1 |\widehat{G}_n(u)| u^{-1} du |\lambda - \widehat{\lambda}_n|.$$

Then, using Assumption 3.1, $(C_{1,2}^\lambda)$ and (C_2^G) , the convergence in probability of \widehat{K}_n towards K follows. The proof of the convergence in probability of \widehat{s}_n towards s in probability is quite similar. \square

We deduce easily from the above proposition that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\|(\widehat{K}_n - K)r\| \geq \varepsilon \right) = 0.$$

Let us choose $\eta > 0$ and $\varepsilon > 0$ such that $\varepsilon < 1 - \|K\|$. We define

$$\Omega_n = \left\{ \omega \in \Omega ; \|\widehat{K}_n(\omega)\| < 1 - \varepsilon, \|(K - \widehat{K}_n(\omega))r\| \leq \frac{\varepsilon^2}{4}, \|s - \widehat{s}_n(\omega)\| \leq \frac{\varepsilon^2}{4} \right\}.$$

According to Proposition B.1, there exists N such that for all $n \geq N$,

$$\begin{aligned} \mathbb{P}(\Omega \setminus \Omega_n) &\leq \mathbb{P} \left(\|\widehat{K}_n\| \geq 1 - \varepsilon \right) + \mathbb{P} \left(\|(K - \widehat{K}_n)r\| > \frac{\varepsilon^2}{4} \right) + \mathbb{P} \left(\|s - \widehat{s}_n\| > \frac{\varepsilon^2}{4} \right) \\ &\leq \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta \end{aligned}$$

From (14), \widehat{p}_n satisfies almost-surely the equation,

$$\widehat{p}_n = \widehat{s}_n + \widehat{K}_n \widehat{p}_n.$$

Therefore, we also have $\hat{p}_n = \sum_{k=0}^{\infty} \hat{K}_n^k \hat{s}_n$. We split the difference $p - \hat{p}_{n,m}$ using the quantity \hat{p}_n ,

$$p - \hat{p}_{n,m} = p - \hat{p}_n + \hat{p}_n - \hat{p}_{n,m}.$$

We begin to bound $p - \hat{p}_n$ on Ω_n . For $n \geq N$, on Ω_n ,

$$\begin{aligned} \|p - \hat{p}_n\| &\leq \|(s - \hat{s}_n)p\| + \|(K - \hat{K}_n)p\| + \|\hat{K}_n\| \|p - \hat{p}_n\| \\ &\leq \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} + (1 - \varepsilon) \|p - \hat{p}_n\|. \end{aligned}$$

An elementary re-arranging yields $\|p - \hat{p}_n\| \leq \frac{\varepsilon}{2}$ on Ω_n . It remains to consider the difference $\hat{p}_n - \hat{p}_{n,m}$. By definition, we have

$$\hat{p}_n - \hat{p}_{n,m} = \sum_{k=m+1}^{\infty} \hat{K}_n^k \hat{s}_n.$$

Therefore, for $n \geq N$, on Ω_n ,

$$\|\hat{p}_n - \hat{p}_{n,m}\| \leq \|\hat{s}_n\| \frac{\|\hat{K}_n\|^{m+1}}{1 - \|\hat{K}_n\|} \leq \left(\frac{\varepsilon^2}{2} + \|s\| \right) \frac{(1 - \varepsilon)^{m+1}}{\varepsilon} \leq \frac{\varepsilon}{2}$$

for $m \geq M$ with M big enough. Therefore, for $n \geq N$ and $m \geq M$, on Ω_n ,

$$\|p - \hat{p}_{n,m}\| \leq \|p - \hat{p}_n\| + \|\hat{p}_n - \hat{p}_{n,m}\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This concludes the proof.

B.5 Proof of Theorem 3.8

All the ingredients for this proof are in fact already present in the proof of Theorem 3.6. Nevertheless, let us give some details. As in the previous section, let us choose $\eta > 0$ and $\varepsilon > 0$ such that $\varepsilon < 1 - \|K\|$. We define

$$\Omega_n = \left\{ \omega \in \Omega ; \|\hat{K}_n(\omega)\| < 1 - \varepsilon, \|(K - \hat{K}_n(\omega))\| \leq \frac{\varepsilon^2}{2\|s\|}, \|s - \hat{s}_n(\omega)\| \leq \frac{\varepsilon}{2} \right\}.$$

According to Proposition B.1, there exists N such that, for all $n \geq N$,

$$\mathbb{P}(\Omega \setminus \Omega_n) \leq \eta.$$

For $m = 1$ we have,

$$\|\hat{t}_{n,1} - t_1\| = \|\hat{s}_n - s\| \leq \frac{\varepsilon}{2}$$

on Ω_n and the result follows. Now for $m \geq 2$, one may write

$$\hat{t}_{n,m} - t_m = \hat{K}_n(\hat{t}_{n,m-1} - t_{m-1}) + (\hat{K}_n - K)t_{m-1}.$$

Notice that on Ω_n , for $m \geq 1$,

$$\|t_m\| = \|K^{m-1}s\| \leq \|K\|^{m-1}\|s\| \leq (1 - \varepsilon)^{m-1}\|s\|.$$

Then, for $m \geq 2$,

$$\begin{aligned} \|\hat{t}_{n,m} - t_m\| &\leq \|\hat{K}_n(\hat{t}_{n,m-1} - t_{m-1})\| + \|(\hat{K}_n - K)t_{m-1}\| \\ &\leq \|\hat{K}_n\| \|\hat{t}_{n,m-1} - t_{m-1}\| + \|(\hat{K}_n - K)\| \|t_{m-1}\| \\ &\leq (1 - \varepsilon) \|\hat{t}_{n,m-1} - t_{m-1}\| + \frac{\varepsilon^2}{2\|s\|} (1 - \varepsilon)^{m-2} \|s\|. \end{aligned}$$

A straightforward recursion gives, always for $m \geq 2$ and on Ω_n ,

$$\|\hat{t}_{n,m} - t_m\| \leq (1 - \varepsilon)^{m-1} \|\hat{t}_{n,1} - t_1\| + \frac{\varepsilon^2}{2} (1 - \varepsilon)^{m-2} \sum_{k=0}^{m-2} (1 - \varepsilon)^k.$$

Therefore, for any $m \geq 2$, on Ω_n ,

$$\|\hat{t}_{n,m} - t_m\| \leq (1 - \varepsilon)^{m-1} \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2} (1 - \varepsilon)^{m-2} \frac{1 - (1 - \varepsilon)^{m-1}}{\varepsilon} \leq \varepsilon.$$

C Discussion on the condition (C_2^G)

Here, we propose to show that the Parzen-Rosenblatt estimator \hat{G}_n^{PR} of the density G , defined by

$$\forall x \in [0, 1], \hat{G}_n^{\text{PR}}(x) = \frac{1}{nh_n} \sum_{i=1}^n \mathbb{K}\left(\frac{Y_i - x}{h_n}\right),$$

where \mathbb{K} is a kernel function and the bandwidth sequence $(h_n)_{n \geq 1}$ tends to 0 as n goes to infinity, satisfies the condition (C_2^G) under the following assumption on the density of interest.

Assumption C.1 *We assume that there exists a real number $\epsilon_1 > 0$ such that, for any $0 \leq x < \epsilon_1$, $G(x) = 0$. In addition, we suppose that G is in the Hölder class $\Sigma(\beta, L)$ (see [20, Definition 1.2]).*

For any x , we define the mean squared error of $\hat{G}_n^{\text{PR}}(x)$ by

$$\text{MSE}(x) = \mathbb{E} \left[\left(\hat{G}_n^{\text{PR}}(x) - G(x) \right)^2 \right].$$

By [20, equation (1.4)], we have the following bias-variance decomposition

$$\text{MSE}(x) = b^2(x) + \sigma^2(x),$$

where, with [20, equation (1.6)],

$$b(x) = \mathbb{E} \left[\hat{G}_n^{\text{PR}}(x) \right] - G(x) \quad \text{and} \quad \sigma^2(x) = \frac{1}{nh_n^2} \mathbb{E} \left[\mathbb{K}^2 \left(\frac{Y_1 - x}{h_n} \right) \right].$$

In the sequel, we assume that the chosen kernel function \mathbb{K} has a bounded support. As a consequence, for n large enough and some $\epsilon_2 > 0$, $\mathbb{K}\left(\frac{y-x}{h_n}\right) = 0$ for any $x < \epsilon_2$ and $y \geq \epsilon_1$. Thus,

$$\int_0^1 \frac{\sigma(x)}{x} dx = \frac{1}{\sqrt{nh_n}} \int_{\epsilon_1}^1 \frac{1}{x} \left[\int_{\epsilon_2}^1 G(y) \mathbb{K}^2 \left(\frac{y-x}{h_n} \right) dx \right]^{1/2} dy \leq -\frac{\|\mathbb{K}\|_{\infty} \log(\epsilon_1)}{\sqrt{nh_n}}. \quad (26)$$

In addition, $b(x) = 0$ for any $x < \epsilon_1 \wedge \epsilon_2$. Therefore, by virtue of [20, Proposition 1.2],

$$\int_0^1 \frac{b(x)}{x} dx = \int_{\epsilon_1 \wedge \epsilon_2}^1 \frac{b(x)}{x} dx \leq C_1 h_n^{\beta}, \quad (27)$$

for some positive number C_1 , whenever \mathbb{K} is a kernel of order $l = \lfloor \beta \rfloor$ (see [20, Definition 1.3]) satisfying

$$\int |u|^\beta \mathbb{K}(u) du < \infty.$$

Finally, by (26) and (27), we have

$$\begin{aligned} \mathbb{E} \left[\int_0^1 \frac{|\hat{G}_n^{\text{PR}}(x) - G(x)|}{x} dx \right] &\leq \int_0^1 \frac{\sqrt{\text{MSE}(x)}}{x} dx \\ &\leq \int_0^1 \frac{b(x)}{x} dx + \int_0^1 \frac{\sigma(x)}{x} dx \\ &\leq C_2 \left(\frac{1}{\sqrt{n}h_n} + h_n^\beta \right), \end{aligned}$$

for some constant C_2 . We conclude that the L^1 -norm vanishes when n tends to infinity if the bandwidth is such that $\sqrt{n}h_n \rightarrow 0$. Therefore, the convergence in probability (C_2^G) holds under this condition.

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Romain Azaïs

Inria Sophia Antipolis Méditerranée, Team Virtual Plants

romain.azais@gmail.com

Alexandre Genadot

Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, Paris 6

algenadot@gmail.com